Orthogonal Basis Vectors for Generating Artifact-free Recording Signals

by C. Bond, 2011

1 Background

In a 1973 paper by Williams and Comstock[1] a method for generating ideal Lorentzian pulse trains was presented. The method was extended in a 2003 paper by the author[2], to support arbitrary patterns typical of those found in magnetic recording. The motivation was to find a strategy for generating test signals which did not suffer from distortion due to windowing, or the truncation required in the superposition of isolated pulses. The resulting signal trains can be seamlessly repeated indefinitely.

The equation found by W&C is:

$$V(x) = \left(\frac{V(0)}{2B}\right) \frac{\pi P \sinh(\pi P/2B) + \cos(\pi x/B)}{\cosh^2(\pi P/2B) - \cos^2(\pi x/B)}, \tag{1}$$

where B is the bit cell length and P is thePW_{50} as a fraction of that length. $V(0)$ is the peak amplitude. The expression was derived using residue calculus.

For reasons of computational efficiency, it would be better to convert the squared functions into equivalent double angle forms,

$$V(x) = \left(\frac{V(0)}{B}\right) \frac{\pi P \sinh(\pi P/2B) + \cos(\pi x/B)}{\cosh(\pi P/B) - \cos(2\pi x/B)}. \tag{2}$$

This paper provides a summary of the extended method for generating arbitrary pattens and a more detailed account of its usage. A further extension

[1]Williams, M.L. and Comstock, R.L.,"Frequency Response in Digital Magnetic Recording," *IEEE Trans. Magn.*, vol. MAG-9, pp. 342-345,Sept. 1973

[2]Bond, C., "An Extensible Set of Orthogonal Functions for Creating Ideal Channel Test Patterns," *IEEE Trans. Magn.*, vol. 39, no.2, pp. 1070-1071, Mar. 2003

supporting a 4th-order pulse model is also presented. This model is a better signal approximation for near field applications than is the Lorentzian.

2 Basis Vectors

Support for arbitrary patterns is made possible by the properties of orthogonal vectors, which provide a means to build composite signals from independent building blocks. In the paper by the author, an extensible set of vectors were defined which apply to typical digital recording patterns. In particular, combinations of elements from a ternary set consisting of $\{0, +1, -1\}$ are chosen consistent with the properties of recording signals.

Recall that during the write process, current in the recording head is switched from one direction to the other. Typically, a $+1$, represents switching the current in one direction and -1 represents switching in the other direction. The resulting sequence of values is divided into *cells* with a single bit occupying each cell. Any number of consecutive 0's is possible, depending on the recording code used, but only one $+1$ or -1 can occur in successive cells before the other value appears. Thus, consecutive $+1$'s or -1's do not occur. Other interpretations of the bit sequence are possible, but we use the convention described above for convenience. Additional signal properties will be discussed later in this paper.

Since every $+1$ is eventually followed by a -1 and vice versa, the basis vectors will consist of an even number of 1's. The simplest vector is $\{+1, -1\}$. Next, we have $\{+1, 0, -1, 0\}$ whose orthogonal counterpart is $\{0, +1, 0, -1\}$. The $\{+1, -1\}$ sequence can be extended to $\{+1, -1, +1, -1\}$ to form a set of three vectors of fixed length. There will be $n - 1$ vectors in a set, where n is the length of the longest sequence. We define a *subset* as a collection of vectors having the same number of zeros.

Each subset will consist of $k/2$ vectors, where $k = 2^p$ is the length of its sequence. The reason we do not require k vectors of length k is that the weighting function which will be assigned to each vector is a signed quantity. Hence, the negative versions of the first $k/2$ vectors will already be available.

Clearly, the small set of vectors previously written satisfies the orthogonal property wherein the sum of products of corresponding elements for any two vectors is zero, except for the product of any vector with itself. That property will hold over the extended set which allows for longer signals.[3]

The basis is extended by adding subsets consisting of $k/2$ vectors with k elements each. A subset starts with a vector having 1 as the first element and -1 as the element in position $k/2$, using 0-based indexing. The other elements in each vector are 0's. Additional members of the set are formed by rotating the elements of the first vector one position at a time until $k/2$ vectors are defined. We will refer to this operations as shifting, since this is conceptually simpler and unambiguous. Since the length of each vector in a subset is twice the length of the vectors in the previous subset, lower order vectors should be extended by duplicating them until they have the same length as the members of the highest subset.

A 16 element basis vector set is shown in Fig.(1) where, for convenience, we have used j to identify the subset and d to represent the delay (number of leading zeros).

3 Signal Generation

The general strategy for creating arbitrary recording signals is to apply linear algebra to the desired ternary signal vector and the matrix formed from the basis vectors in order to find the weighting vector. That is,

$$\mathbf{A} \cdot \mathbf{v} = \mathbf{b}, \tag{3}$$

where $\mathbf{A}$ is the matrix of basis vectors, $\mathbf{v}$ is the unknown weighting function for the associated basis vectors, and $\mathbf{b}$ is the desired signal.

The weighting vector $\mathbf{v}$ represents the contribution of each basis vector to the composite signal. Each element of $\mathbf{v}$ multiplies a column of $\mathbf{A}$ so the basis vectors must be arranged in columns. This is done by transposing the array in Fig.(1).

[3] The basis can also be made orthonormal by scaling the elements with $1/\sqrt{2}$, but there is no need for orthonormality in this context.

$\mathcal{S}(j,d)$	sequence															
$\mathcal{S}(0,0)$	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
$\mathcal{S}(1,0)$	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0
$\mathcal{S}(1,1)$	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1
$\mathcal{S}(2,0)$	1	0	0	0	-1	0	0	0	1	0	0	0	-1	0	0	0
$\mathcal{S}(2,1)$	0	1	0	0	0	-1	0	0	0	1	0	0	0	-1	0	0
$\mathcal{S}(2,2)$	0	0	1	0	0	0	-1	0	0	0	1	0	0	0	-1	0
$\mathcal{S}(2,3)$	0	0	0	1	0	0	0	-1	0	0	0	1	0	0	0	-1
$\mathcal{S}(3,0)$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
$\mathcal{S}(3,1)$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
$\mathcal{S}(3,2)$	0	0	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$\mathcal{S}(3,3)$	0	0	0	1	0	0	0	0	0	0	0	-1	0	0	0	0
$\mathcal{S}(3,4)$	0	0	0	0	1	0	0	0	0	0	0	0	-1	0	0	0
$\mathcal{S}(3,5)$	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	0	0
$\mathcal{S}(3,6)$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	0
$\mathcal{S}(3,7)$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1

Figure 1: 16-Element Basis Vectors

A reduction in the size of the matrix can be justified by noting that a k length sequence of recording signals must be of even parity. Thus, the sum of all ternary elements will be zero. This means that only $k-1$ elements can be independently specified, because the last element will necessarily have a value that forces the sum to zero. Otherwise the sequence cannot be duplicated endlessly.

The desired signal will consist of 2^p elements or bits, although only $2^p - 1$ need be specified. Hence, $\mathbf{A}$ is a $k-1$ by $k-1$ matrix. The matrix corresponding to $p = 3$, and $k = 8$ is the 7 by 7 array shown in (4), suitably transposed.

$$\mathbf{A} = \left|\begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \end{array}\right| \tag{4}$$

A modified form of Eq.(1) will be required to support vectors with varying numbers of zeros but with the same PW_{50}. Such an equation is given in Eq.(5).

$$V(x, v, b, d) = \left(\frac{v}{b}\right) \frac{\pi P \sinh(\pi P/2b) + \cos(\pi(x-d)/b)}{\cosh(\pi P/b) - \cos(2\pi(x-d)/b)} \tag{5}$$

where P is set globally, b varies with the length of the basis vector sequence, x is the distance along the track, and d is the delay of the basis vector within its subset. The amplitude v is the associated element from the weighting vector $\mathbf{v}$; *i.e.* v_m is the mth element of $\mathbf{v}$.

To map appropriate values from the problem to the solution, we require that $b = 2^j$ and $m = b + d - 1$, where $S(j, d) \mapsto V(x, v, b, d)$. In this case, b represents an extended bit cell corresponding to the length of the basic pattern in the subset.

Once the weighting vector is found, we will sum the derived values of Eq. (5) from which the analog test signal can be generated at any convenient resolution.[4]

4 Data Recording Examples

In this section we will examine a very simple signal example in detail. Plots of more elaborate signals will be given for reference.

Suppose we want to generate a string of dibits. The signal will consist of $k = 2^p$ bits as previously stated, and for this short sequence $\{1, -1, 0\}$ we

[4]Resolution, in this context, refers to the number of samples per bit cell.

find $k = 4$ so $p = 2$. Of course, we only need to specify $k - 1$ bits, so the problem statement is complete. Our solution sequence will actually be: $\{1, -1, 0, 0\}$.

The matrix equation looks like this:

$$\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} \begin{vmatrix} v_0 \\ v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}, \tag{6}$$

from which the solution vector $\mathbf{v} = \{1/2, 1/2, -1/2\}$.

The signal is generated by executing the following summations:

$$P = \sum_{j=0}^{k-1} \sum_{d=0}^{j} V(x, v_m, b, d), \tag{7}$$

where, as previously stated, $m = b + d - 1$ and $b = 2^j$.

A plot of the result is shown in Fig. (2).

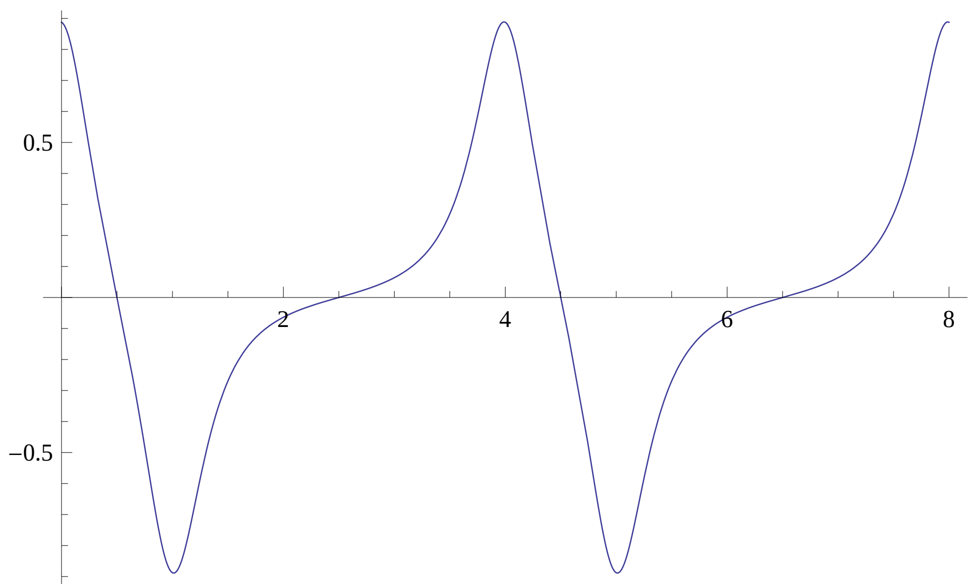

Figure 2: Simple Dibit Test Pattern

Here is a repeating tribit pattern with the bitcell equal to one unit, $B = 1$, and $PW_{50} = 0.7$.

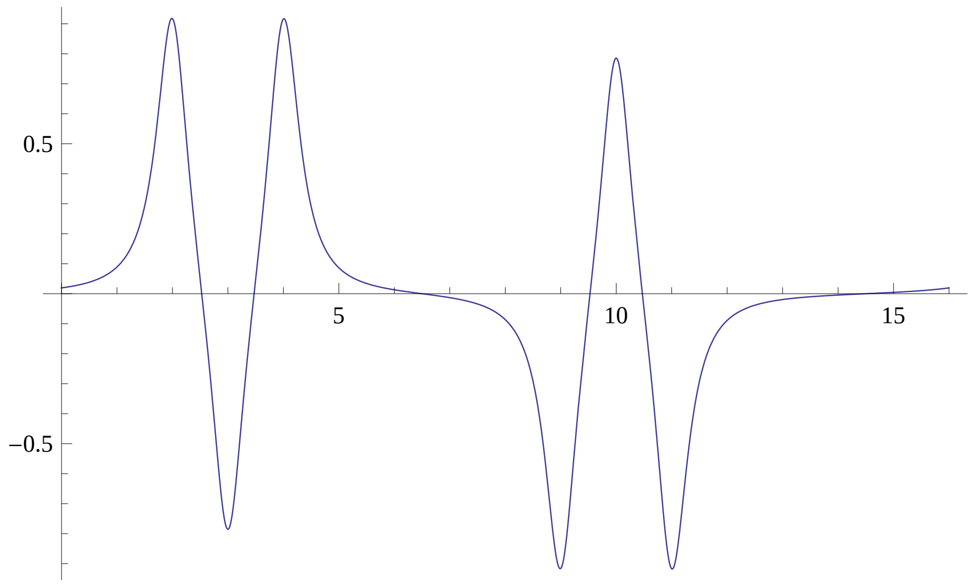

Figure 3: Repeating Tribit Pattern

Here is a random signal vector with $B = 2$ and $P = 1.5$.

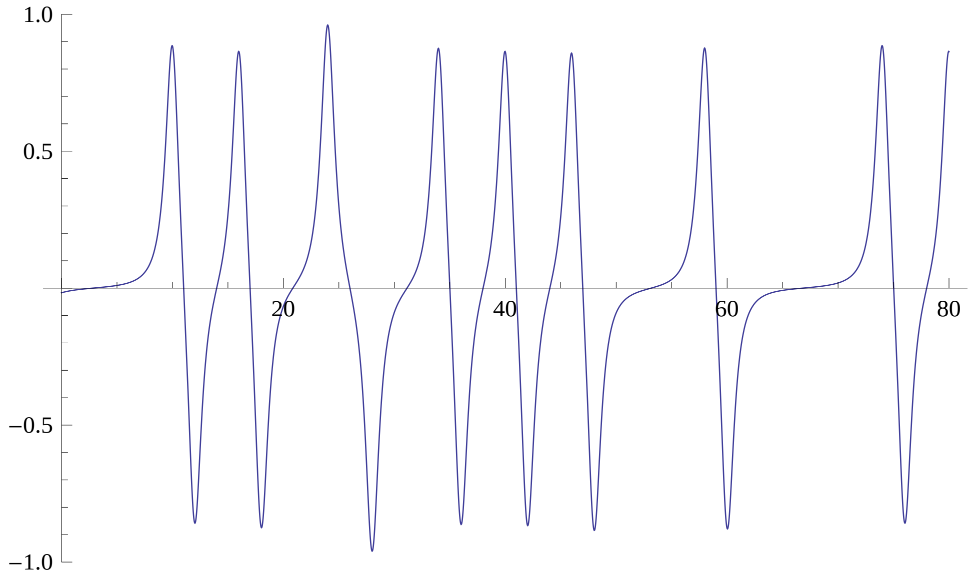

Figure 4: Random Signal Vector

Servo Examples The previous exposition implies that the possible signal patterns must consist of alternating 1's with 0's in arbitrary positions. It happens that the method is more general than that and can reproduce more complex signals as long as polarities alternate and the parity of the ternary specification is zero.

Here is an example of a possible servo signal which satisfies our requirement for ideal, artifact-free properties.

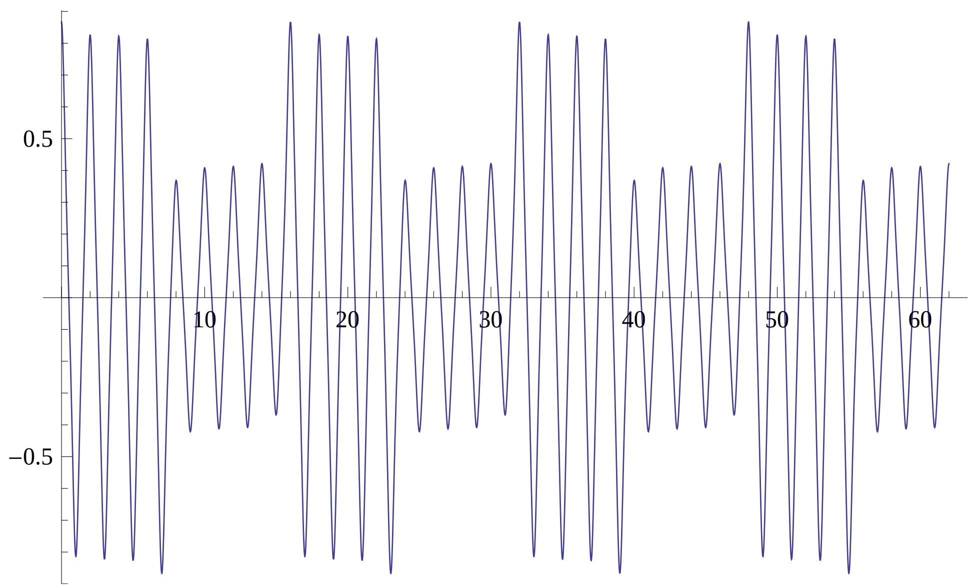

Figure 5: A-B Servo Bursts

Track Interference Examples We can also alter the basic desired signal by adding interference as show in Fig.(6).

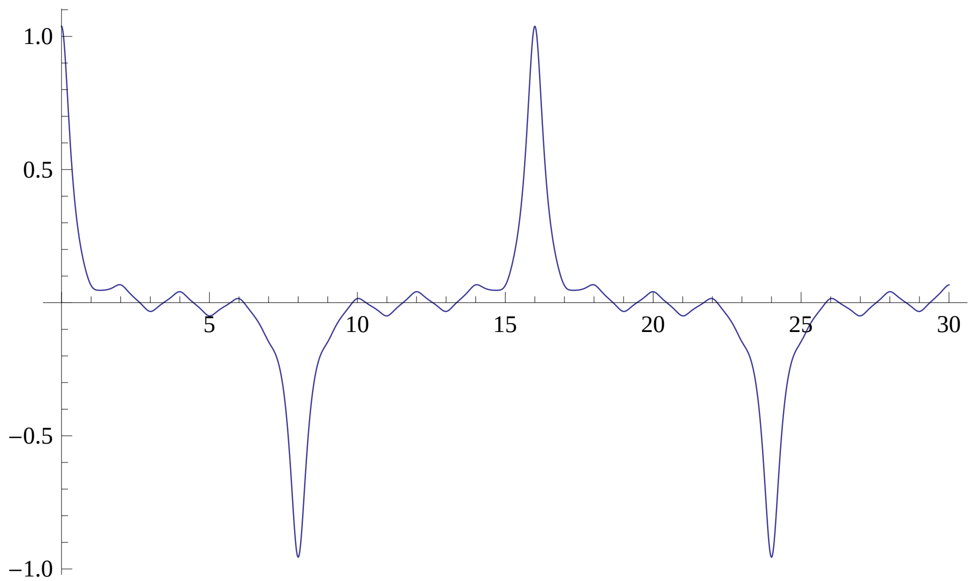

Figure 6: On-track Signal with Coherent Off-track Interference

5 A Fourth Order Pulse Extension

5.1 Lorentzian Pulses

It is a common practice to model recording signals with so-called Lorentzian pulses. The simplest mathematical form for the Lorentzian is:

$$\mathcal{L}(x) = \frac{1}{1+x^2} \tag{8}$$

where x is the unscaled distance along a recorded track. The shape is shown in Fig. (7).

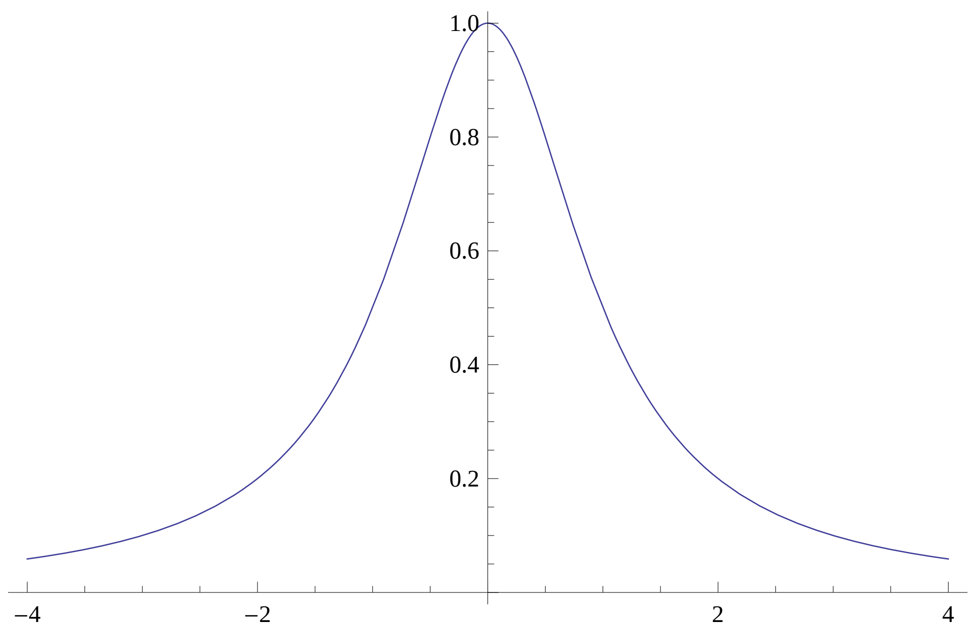

Figure 7: Lorentzian Pulse

One advantage of the Lorentzian is that it is the derivative of an arctangent function which is a fair approximation to the magnetic field near a recorded transition. In Fig. (8) a pulse and its (scaled) integral are shown for comparison.

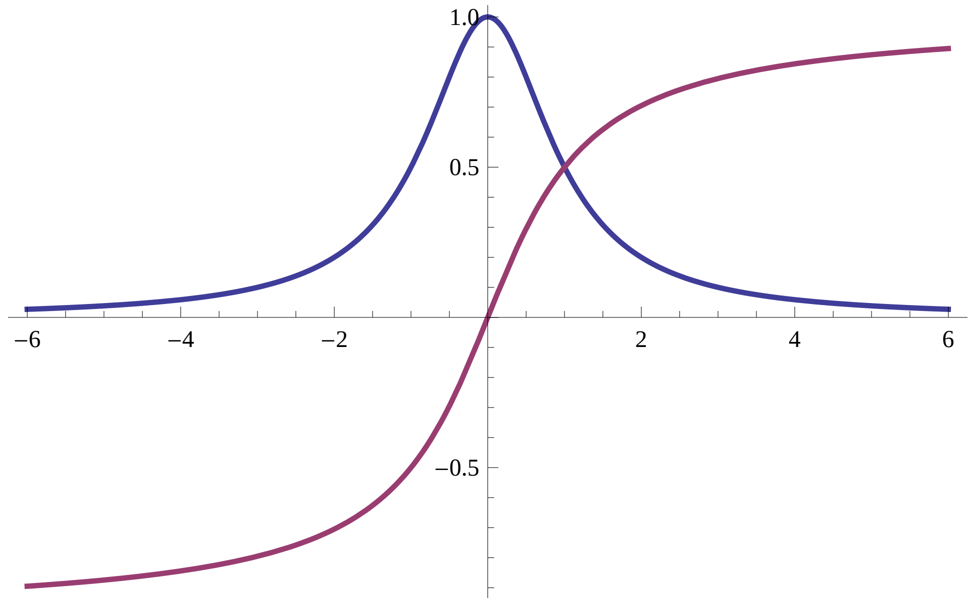

Figure 8: Lorentzian Pulse with Arctangent

A disadvantage of this pulse shape is that it has 'skirts' which decrease rather slowly with distance from the center. Consequently, it is a better approximation for medium to far field analysis than for near field, where 'near' and 'far' loosely refer to whether the head is close to the media or separated by some distance.

5.2 Gaussian Pulse

When the fly height is very small, a pulse model with more rapidly converging skirts is more appropriate. One such model is the Gaussian pulse which can be expanded as follows:

$$\mathcal{G}(x) = e^{-x^2} = \frac{1}{1 + x^2 + x^4/2 + x^6/6 + x^8/24 + \cdots}. \tag{9}$$

Note that retaining only the first two terms in the denominator polynomial will produce the Lorentzian pulse. Addition of more terms accelerates the convergence of the skirts to the axis. Even one additional term significantly affects the shape, as can be seen in Fig. (9), where their shapes are compared. The expressions for each of these pulses has been modified by a scaling factor

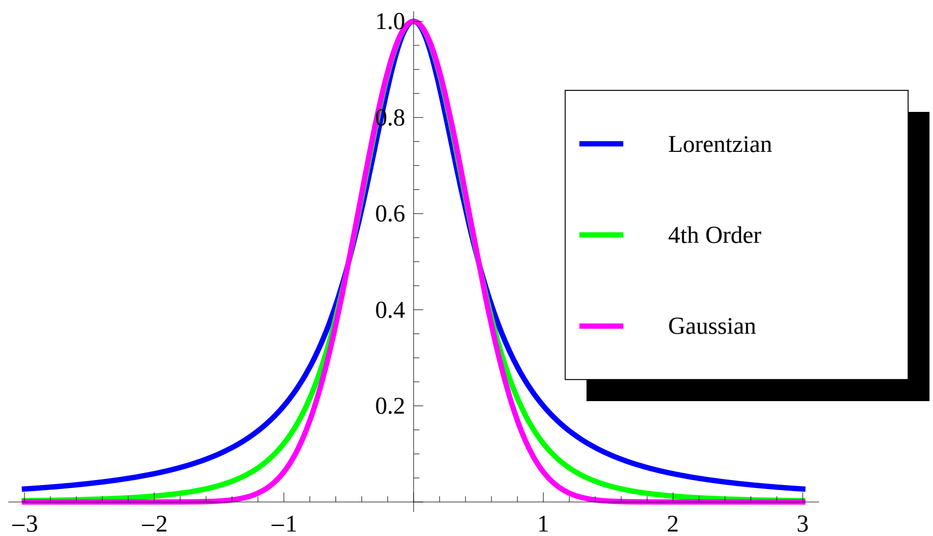

Figure 9: Lorentzian, Intermediate (4th Order) and Gaussian Pulses

so that the PW_{50} occurs where $x = \pm 0.5$. Specifically, the Lorentzian is scaled by replacing x with $2x$. The intermediate pulse, using three terms of the expansion of e^{-x^2} is scaled by replacing x with $2\sqrt{\sqrt{3}-1}\,x$. Finally, the Gaussian pulse is scaled by replacing x with $2\sqrt{\log(2)}\,x$.

5.3 Generalized, Symmetric Pulse Function

We propose to define a family of pulse shapes which cover the range between a Lorentzian and Gaussian shape. It can be expressed as:

$$\mathcal{P}_n(x) = \left(\sum_{k=0}^{n} \frac{x^{2k}}{k!} \right)^{-1} . \tag{10}$$

where x is the horizontal displacement or time, and n is a parameter used to select specific members of the pulse family.

Note that the *unnormalized* far spacing (Lorentzian) and near spacing (Gaussian) pulse models are special cases of the function for which $n = 1$ and $n = \infty$, respectively. That is,

$$\mathcal{P}_1(x) = \mathcal{L}(x) = \frac{1}{1+x^2} \quad (11)$$

$$\mathcal{P}_2(x) = \frac{1}{1+x^2+x^4/2} \quad (12)$$

$$\mathcal{P}_\infty(x) = \mathcal{G}(x) = \frac{1}{1+x^2+x^4/2!+\cdots+x^k/k!+\cdots} = e^{-x^2}, \quad (13)$$

so our model is $\mathcal{P}_2(x)$.

Just as W&C scaled the Lorentzian with an adjustable PW_{50} control parameter, we can scale $\mathcal{P}_2(x)$ with

$$\mathcal{B}(x) = \frac{1}{1+(k\,x/P)^2+(k\,x/P)^4/2} \quad (14)$$

where P is the PW_{50} and k is a scaling parameter such that

$$\mathcal{B}(1/2) = 1/2 \quad (15)$$

when $P = 1$. For our pulse $k = 2\sqrt{\sqrt{3}-1} = 1.7112$.

Following the exposition at the first part of this paper, we can adapt the pulse for artifact-free signal modeling. This is done by defining the following:

$$p = \frac{1}{1+\left(k(Bz+(x-d))/P\right)^2+\frac{1}{2}\left(k(Bz+(x-d))/P\right)^4}, \quad (16)$$

where B is the number of bit cells between pulses and z is a complex variable associated with the poles of the function. The delay parameter, d was previously defined for Eq. (5).

The task of finding the residues at the poles of this function is much more complicated than in the Lorentzian case. The computer is your friend here, and can find a solution in less time than it takes to type in the problem.

Comparison of Models A comparison of pulse sequences formed from this shape with Lorentzians reveals only slight difference, but the difference may be important for demanding modeling projects. In the following comparison plots the Lorentzian is colored *blue* and the 4th order pulses are *magenta*.

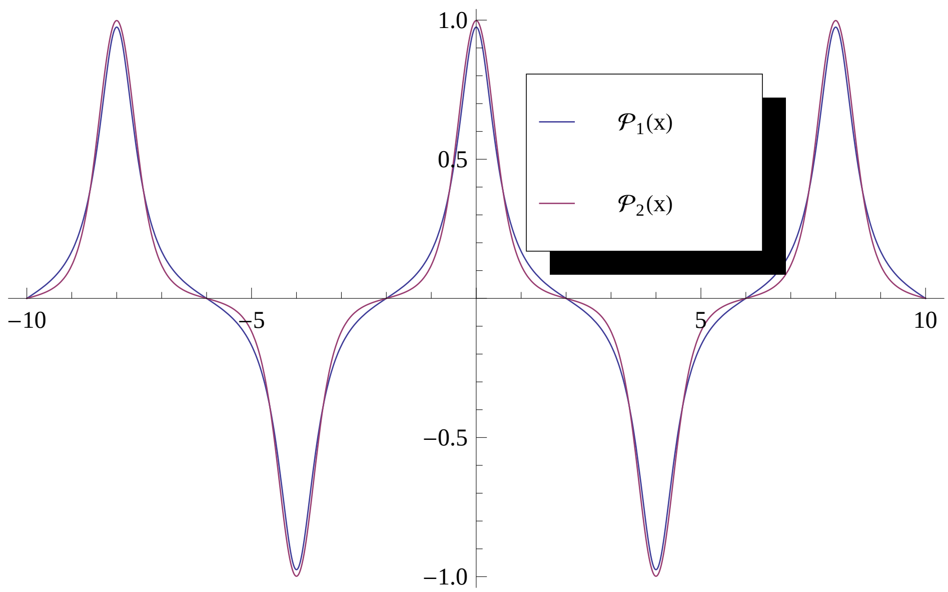

Figure 10: Comparison of Lorentzian and 4th Order Pulse Trains

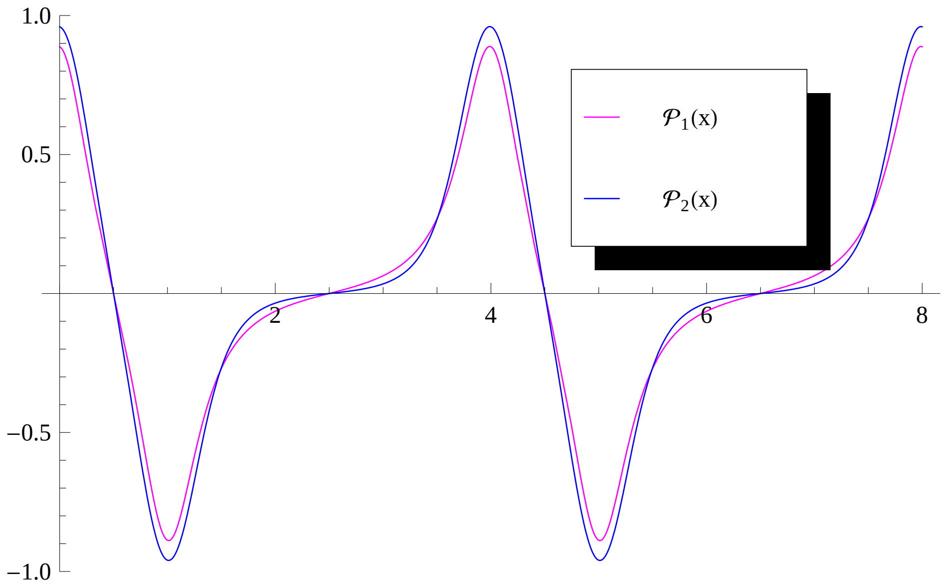

Figure 11: Comparison of Lorentzian and 4th Order Dibits

Random Signal Here is a random sequence comparing Lorentzian and 4th order models, ($\mathcal{P}_1(x)$ and $\mathcal{P}_2(x)$).

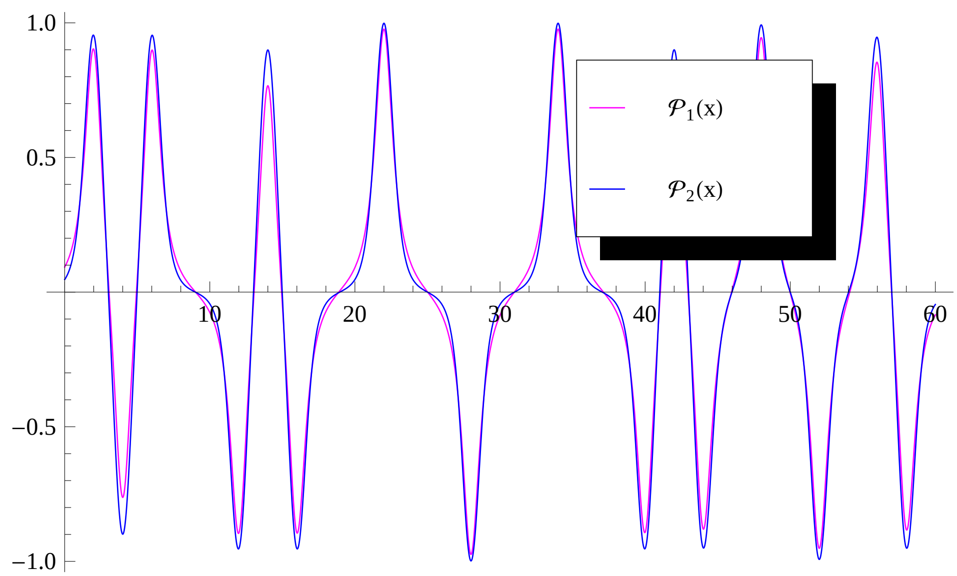

Figure 12: Comparison of Random Signals

Whether the 4th order pulse properties have any significant advantage over other shapes will certainly be application dependent. Nevertheless, most technologists like to keep as many tools at hand as they can manage.